



University of
New Haven

University of New Haven
Digital Commons @ New Haven

Mathematics Faculty Publications

Mathematics

8-2018

Gradient Ricci Solitons with a Conformal Vector Field

Ramesh Sharma

University of New Haven, rsharma@newhaven.edu

Follow this and additional works at: <https://digitalcommons.newhaven.edu/mathematics-facpubs>



Part of the [Mathematics Commons](#)

Publisher Citation

Sharma, R. (2018). Gradient Ricci solitons with a conformal vector field. *Journal of Geometry*, 109(2), 33.

Comments

This is a post-peer-review, pre-copyedit version of an article published in *Journal of Geometry*. The final authenticated version is available online at:

<http://dx.doi.org/10.1007/s00022-018-0439-x>.

Mathematics Subject Classification 53C25 53C44

GRADIENT RICCI SOLITONS WITH A CONFORMAL VECTOR FIELD

Ramesh Sharma

Abstract

We show that a connected gradient Ricci soliton (M, g, f, λ) with constant scalar curvature and admitting a non-homothetic conformal vector field V leaving the potential vector field invariant, is Einstein and the potential function f is constant. For locally conformally flat case and non-homothetic V we show without constant scalar curvature assumption, that f is constant and g has constant curvature.

2010 *Mathematics Subject Classification*: 53C25, 53C44

Keywords and phrases: Gradient Ricci soliton, constant scalar curvature, conformal vector field.

1 Introduction

Let M denote a smooth n -dimensional manifold, g a Riemannian metric and X a smooth vector field on M , and λ a real constant. Then the system (M, g, X, λ) is said to define a Ricci soliton if

$$L_X g + 2 \operatorname{Ric} = 2\lambda g \quad (1)$$

where L denotes the Lie-derivative operator and Ric the Ricci tensor of g . Thus a Ricci soliton is a generalization of an Einstein metric for which X is Killing. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero, and negative respectively. If the vector field X is the gradient of a smooth function f , i.e. $X = \nabla f$, then (M, g, f, λ) is called a gradient Ricci soliton, in which case the equation (1) becomes

$$\operatorname{Hess} f + \operatorname{Ric} = \lambda g \quad (2)$$

where Hess denotes the Hessian operator with respect to g . An important result of Perelman [9] says that a compact Ricci soliton is gradient. The gradient Ricci soliton is said to be trivial when f is constant and g is Einstein.

For a general Ricci soliton vector field X , we have the following formula (Chow et al [1]):

$$L_X S = 2|\text{Ric}|^2 + \Delta S - 2\lambda S \quad (3)$$

for the scalar curvature S , where $\Delta = \text{Tr} \cdot (\text{Hess})$ denotes the Laplacian operator of g .

In [3], Fernández-López and García-Río showed that conformally flat gradient Ricci solitons are locally isometric to a warped product of an interval and a real space form. This result was generalized to include the Lorentzian case by Brozos-Vázquez, García-Río and Gavino-Fernández in [2]. We also note that a Riemannian n -manifold admitting a maximal $\frac{(n+1)(n+2)}{2}$ -parameter group of conformal transformations is conformally flat. Therefore it is interesting to examine the effect of the existence of a 1-parameter group of conformal transformations generated by a conformal vector field V on a gradient Ricci soliton. Motivated by this problem, we prove

Theorem 1 *If (M, g, f, λ) is a connected gradient Ricci soliton with constant scalar curvature and admits a non-homothetic conformal vector field V leaving the potential vector field ∇f invariant, then g is Einstein and the potential function f is constant.*

Remark 1. Theorem 1 was motivated by a similar result of Jauregui and Wylie [5]: “A gradient Ricci soliton admitting a non-homothetic conformal vector field V that preserves the gradient 1-form df (i.e. $\nabla_V f$ is constant) is Einstein and f is constant”. We note that the hypothesis “ $\nabla_V f$ is constant” in the result of Jauregui and Wylie, does not imply the hypothesis “ V leaves the potential vector field ∇f invariant” of Theorem 1. For f constant, g is Einstein (scalar curvature is obviously constant) for which Yano and Nagano [12] proved: “A complete Einstein manifold admitting a complete non-homothetic conformal vector field is isometric to a round sphere.” However, if only M is complete and V not necessarily complete, then by a result of Kanai [6] (stated also in Kühnel and Rademacher [7]), M is isometric to one of the following spaces: S^n , E^n , H^n , the warped product $R \times_{\exp} M_*$ where (M_*, g_*) is complete and Ricci flat, or the warped product $R \times_{\cosh} M_*$ where (M_*, g_*) is complete and Einstein with $S_* = -1$.

Remark 2. Constant scalar curvature gradient Ricci Solitons were studied by Petersen and Wylie [10] who showed that a shrinking (respectively,

expanding) gradient Ricci soliton with constant scalar curvature S satisfies $0 \leq S \leq n\lambda$ (respectively, $n\lambda \leq S \leq 0$). Also, g is flat if $S = 0$ and Einstein when $S = n\lambda$. Fernández-López and García-Río [4] showed that, if an n -dimensional complete gradient Ricci soliton has constant scalar curvature S then $S \in \{0, \lambda, \dots, (n-1)\lambda, n\lambda\}$. Thus the problem of classifying gradient Ricci solitons with constant scalar curvature is, in general, open.

For the case when V is homothetic, we prove

Proposition 1 *If (M, g, f, λ) is a gradient Ricci soliton with a homothetic vector field V leaving the potential vector field ∇f invariant, then either (i) it is a Gaussian soliton, or (ii) V is Killing. In case (ii), either the soliton is steady or V preserves f .*

A conformal vector field V on a Riemannian manifold (M, g) is defined by

$$L_V g = 2\sigma g \quad (4)$$

where σ is a smooth function on M . V is homothetic when σ is constant, and is Killing when $\sigma = 0$. Denoting the Riemannian connection as well as the gradient operator of g by ∇ we have the following formula:

$$(L_V \nabla)(Y, Z) = (Y\sigma)Z + (Z\sigma)Y - g(Y, Z)\nabla\sigma \quad (5)$$

where Y, Z denote arbitrary smooth vector fields on M . We will follow this notation in the next section.

2 Proofs Of Theorem 1 and Proposition 1

Proof Of Theorem 1. A straightforward computation using the definition (2) provides

$$R(Y, Z)\nabla f + (\nabla_Y Q)Z - (\nabla_Z Q)Y = 0 \quad (6)$$

where R denotes the curvature tensor and Q the Ricci tensor of type (1,1) such that $\text{Ric}(Y, Z) = g(QY, Z)$. Let (e_i) ($i = 1, \dots, n$) be a local orthonormal frame on (M, g) . Substituting e_i for Y in (6), taking inner product with e_i , summing over i , and using the twice contracted second Bianchi identity: $\text{div}(Q) = \frac{1}{2}dS$ yields the known formula

$$Q(\nabla f) = \frac{1}{2}\nabla S \quad (7)$$

Next, differentiating $|\nabla f|^2$ along an arbitrary vector field, and using equations (2) and (7) gives the known formula

$$|\nabla f|^2 + S - 2\lambda f = c \quad (8)$$

where c is a real constant. As S is constant by hypothesis, equation (7) reduces to

$$Q(\nabla f) = 0. \quad (9)$$

At this point, Lie differentiating the relation: $df = g(\nabla f, \cdot)$ along the conformal vector field V , noting that Lie derivative commutes with exterior derivative d , and using the hypothesis $L_V \nabla f = 0$, we find $d(L_V f) = 2\sigma df$. Applying d on it and using the Poincaré lemma: $d^2 = 0$ we obtain

$$(d\sigma) \wedge (df) = 0. \quad (10)$$

Let us now express equation (2) in the form

$$\nabla_Y \nabla f + QY = \lambda Y$$

Taking its Lie derivative along V , using the commutation formula (see [11])

$$L_V \nabla_Y Z - \nabla_Y L_V Z - \nabla_{[V,Y]} Z = (L_V \nabla)(Y, Z)$$

with the choice $Z = \nabla f$, along with the hypothesis $L_V \nabla f = 0$ and equations (2) and (5) yields

$$(L_V Q)Y = -g(\nabla f, \nabla \sigma)Y. \quad (11)$$

Now we substitute e_i for Y in (11), take inner product with e_i , sum over i , and use the constant scalar curvature hypothesis in order to obtain

$$g(\nabla f, \nabla \sigma) = 0 \quad (12)$$

The equations (10) and (12) show that

$$(d\sigma \wedge df)(\nabla \sigma, \nabla f) = |\nabla \sigma|^2 |\nabla f|^2 = 0$$

i.e.

$$|\nabla \sigma| |\nabla f| = 0. \quad (13)$$

As σ is not constant on M , $\nabla \sigma \neq 0$ on an open subset \mathcal{U} of M . So, from (13), $\nabla f = 0$ on \mathcal{U} . Now the g -trace of (2) is $\Delta f + S = n\lambda$ on M . Since

$\Delta f = 0$ on \mathcal{U} , we have $S = n\lambda$ on \mathcal{U} . By hypothesis, S is constant on M and M is connected, and therefore $S = n\lambda$ on M . Using equation (3) with $X = \nabla f$ gives $|\text{Ric}|^2 = \lambda S$. Hence the identity: $|\text{Ric} - \frac{S}{n}g|^2 = |\text{Ric}|^2 - \frac{S^2}{n}$ provides $\text{Ric} = \lambda g$, i.e. g is Einstein. Thus equation (2) reduces to $\nabla \nabla f = 0$, which implies that $|\nabla f|$ is constant. As $\nabla f = 0$ on \mathcal{U} and M is connected, we conclude that $\nabla f = 0$ on M , and so f is constant on M , completing the proof.

Proof Of Proposition 1. Here we have equation (4) with constant σ . Writing equation (2) as

$$L_{\nabla f}g + 2\text{Ric} = 2\lambda g,$$

Lie-differentiating it along V and noting that a homothetic vector field preserves the Ricci tensor we get

$$L_V L_{\nabla f}g = 4\lambda \sigma g$$

Using the identity $L_Y L_Z - L_Z L_Y = L_{[Y, Z]}$ and hypothesis $[V, \nabla f] = 0$ in the above equation we find

$$\sigma(L_{\nabla f}g - 2\lambda g) = 0$$

Hence, either (i) $L_{\nabla f}g - 2\lambda g = 0$, or (ii) $\sigma = 0$. Equation in (i) is basically $\nabla \nabla f = \lambda g$, and by a result (Theorem 2, IB) of Okumura [8]), implies that g is flat and hence is a Gaussian soliton. In case (ii), V is Killing and hence $L_V S = 0$. Also, Lie-differentiating (8) along V and noting that $L_V \nabla f = 0$ and $L_V g = 0$ imply $L_V |\nabla f|^2 = 0$ we find that either $\lambda = 0$ or V preserves f . This completes the proof.

3 Conformally Flat Case

Finally, taking into account the result of [3] for a locally conformally flat gradient Ricci soliton as stated in Section 1, we examine this case with the hypothesis $L_V \nabla f = 0$ of Theorem 1, and without constant scalar curvature assumption and prove

Proposition 2 *If (M, g, f, λ) is a locally conformally flat gradient Ricci soliton and admits a non-homothetic conformal vector field V leaving with the potential vector field ∇f invariant, then f is constant and (M, g) has constant curvature.*

Proof. If f is constant, then we are done. So, $\nabla f \neq 0$ on a neighborhood of some point in M . By a result of [3] we know that (M, g) is locally the warped product of an interval I and an $(n - 1)$ dimensional manifold N of constant curvature c with metric $g = dt^2 + \psi^2(t)\gamma$, where t is the coordinate on I and ψ is the warping function. Also, f is a function of t . The gradient Ricci soliton equation (2) yields (as mentioned in [2])

$$\ddot{f} = \lambda + (n - 1)\frac{\ddot{\psi}}{\psi} \quad (14)$$

$$\psi\dot{\psi}\dot{f} = \lambda\psi^2 - (n - 2)c + \psi\ddot{\psi} + (n - 2)(\dot{\psi})^2 \quad (15)$$

where an over-dot denotes partial differentiation with respect to t . Let us decompose the conformal vector field V on M as $V = \alpha\partial_t + U^k\partial_k$ where α and U^k depend on t as well as the coordinates x^i on N . The components of conformal Killing equation (4) provide

$$\dot{\alpha} = \sigma \quad (16)$$

$$\partial_i\alpha = -(\partial_t U^k)g_{ik} \quad (17)$$

$$L_U g_{ij} = 2(\sigma - \alpha\frac{\dot{\psi}}{\psi})g_{ij} \quad (18)$$

where $U = U^k\partial_k$. The hypothesis: $L_V \nabla f = [V, \nabla f] = 0$ shows

$$\dot{f}\dot{\alpha} = \alpha\ddot{f} \quad (19)$$

$$\partial_t U^k = 0.$$

Hence $U^k = U^k(x^i)$ and equation (17) implies $\alpha = \alpha(t)$. Equation (19) integrates to $\alpha = \dot{f}$ (up to a constant multiple which can be taken 1). Consequently, (16) assumes the form

$$\sigma = \ddot{f} \quad (20)$$

Equation (18) shows that U is homothetic on (N, γ) , i.e. $L_U \gamma = 2k\gamma$ where k is constant such that

$$\ddot{f} - \dot{f}\frac{\dot{\psi}}{\psi} = k. \quad (21)$$

Since (N, γ) has constant curvature c , ${}^\gamma \text{Ric} = c(n-2)\gamma$. Lie-differentiating it along U provides $ck = 0$. This gives rise to two cases (i) $c = 0$, (ii) $k = 0$. For case (i) equations (14), (15) and (21) give us

$$\frac{\ddot{\psi}}{\dot{\psi}} - \frac{(\dot{\psi})^2}{\psi^2} = \frac{k}{n-2} \quad (22)$$

which integrates to $\frac{\dot{\psi}}{\psi} = \frac{k}{n-2}t + a$ and further to $\psi = e^{\frac{k}{n-2}t^2 + at + b}$ where a, b are arbitrary constants. Using (22) in (15) and differentiating with respect to t we get

$$\ddot{f} = \frac{k}{n-2} \left[n-1 - \left(\lambda + \frac{k}{n-2} \right) \left(\frac{k}{n-2}t + a \right)^{-2} \right] \quad (23)$$

Comparing it with (14) we get the polynomial equation

$$(n-1)(Kt+a)^4 + \lambda(Kt+a)^2 + K(\lambda+K) = 0.$$

where $K = \frac{k}{n-2}$. The above equation implies that $k = 0$. Hence (23) reduces to $\ddot{f} = 0$, and from (16) we get $\sigma = 0$ contradicting the non-homotheticity of V . Now we examine the case (ii) $k = 0$ for which (21) integrates to $\dot{f} = \dot{\psi}$. Using this in (14) we have

$$\ddot{\psi} = \frac{\psi}{n-1}(\dot{\psi} - \lambda) \quad (24)$$

Combining this with (15) provides $\psi^2\dot{\psi} = \lambda\psi^2 + (n-1)(\dot{\psi})^2 - (n-1)c$. Differentiating it with respect to t and using (24) gives $\psi^2(\dot{\psi} - \lambda) = 0$. But $\psi \neq 0$ for any t (as g is positive-definite), and so $\dot{\psi} = \lambda$. As already found, $\dot{f} = \dot{\psi}$. Thus $\ddot{f} = \lambda$ and so from (20) we conclude that $\sigma = \lambda$ contradicting the non-homotheticity of V . This completes the proof.

4 Concluding Remark

The assumption $[V, \nabla f] = 0$ in Theorem 1 and Proposition 2 is needed in the proofs, and is trivially satisfied for constant f in which case g is Einstein.

5 Acknowledgment

The author is thankful to the referee for a valuable question.

References

- [1] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, *The Ricci flow: Techniques and Applications, Part I. Geometric Aspects*, Mathematical Surveys and Monographs, **135**, American Mathematical Society, Providence, 2007.
- [2] M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández, *Locally conformally flat Lorentzian gradient Ricci solitons*, J. Geom. Anal. **23** (2013), 1196–1212.
- [3] M. Fernández-López and E. García-Río, *A note on locally conformally flat gradient Ricci solitons*, Geom. Dedicata **168** (2014), 1–7.
- [4] M. Fernández-López and E. García-Río, *On gradient Ricci solitons with constant scalar curvature*, Proc. Amer. Math. Soc. **144** (2016), 369–378.
- [5] J.L. Jauregui and William Wylie, *Conformal diffeomorphisms of gradient Ricci solitons and generalized quasi-Einstein manifolds*, J. Geom. Anal. **25** (2015), 668–708.
- [6] M. Kanai, *On a differential equation characterizing a Riemannian structure of a manifold*, Tokyo J. Math. **6** (1983), 143–151.
- [7] W. Kühnel and H.-B. Rademacher, *Conformal vector fields on pseudo-Riemannian spaces*, Diff. Geom. Appl. **7** (1997), 237–250.
- [8] M. Okumura, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117**(1965), 251–275.
- [9] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, <http://arXiv.org/abs/math.DG/02111159>.
- [10] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol. **14** (2010), 2277–2300.
- [11] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York, 1970.
- [12] K. Yano and T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformations*, Ann. Math. (2) **69** (1959), 451–461.

UNIVERSITY OF NEW HAVEN, WEST HAVEN, CT 06516, USA

E-mail address: rsharma@newhaven.edu